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A Class of Chain Type Estimators for Population Mean Under Two Phase Sampling Using Available Information on Second Variable

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ABSTRACT
We have considered a class of modified ratio-product estimators for finite population mean which make use of coefficient of skewness, coefficient of kurtosis and coefficient of variation of first auxiliary variables. The proposed estimators have been compared with conventional ratio estimator, product type estimators. The results obtained has been demonstrated with numerical illustration carried over the data set of natural population. Suitable recommendation has been put forward to the survey statistician for application in real life population.

KEYWORDS
Ratio estimator, product estimator, finite population mean, double sampling, study variable, auxiliary variable, chain-type, regression type, coefficient of skewness, coefficient of kurtosis, coefficient of variation, bias, mean squared error.

Introduction

It is well established in sample surveys that auxiliary information is often used to improve the precession of estimators of the population parameters. The use of auxiliary information at the estimation stage appears to have started with the work of Cochran (1940). He developed the ratio estimator to estimate the population mean or total of the study variable by using supplementary information on auxiliary variable, positively correlated with study variable. When the auxiliary variable is negatively correlated with the study variable, Robson (1957) proposed the product estimator of the population mean. Several authors including Murthy (1964), Singh and Espejo (2003) etc. contributed a lot in the field of estimation of population parameters in sample surveys through development of ratio and regression type estimators.

It may also be noted that a number of sampling strategies utilize the advance information about an auxiliary variable. When such information is lacking, it is sometimes relatively cheap to take a large preliminary sample in which auxiliary variable alone is measured. The aim of this sample is to obtain a good estimate of the population mean or total of the auxiliary variable or it frequency distributions (Prabhu - Ajgaonkar,

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1975). This technique is known as double sampling or twophase sampling. The two-phase sampling happens to be a powerful and cost effective (economical) technique for obtaining the reliable estimate in first phase sample.

In order to construct an efficient estimator of the population mean of the auxiliary variable in first-phase (preliminary) sample, Chand (1975) gave a technique of chaining another auxiliary variable (which is highly correlated with first auxiliary variable but remotely correlated with the study variable) with the first auxiliary variable by using the ratio estimator in the first phase sample. The estimator is known as chain ratio-type estimator. Further, this work was extended by Kiregyera (1980, 1984), Srivastava (1970, 1990), Singh and Singh(1991), Singh et al.(1994), Singh and Upadhyaya (1995), Singh et al.(2000), Upadhyaya and Singh (2001) and many others by proposing several chain-type ratio and regression estimators.

For making the estimates more precise, Searls (1964) used the coefficient of variation (CV) of study variable at estimation stage. Motivated by Searls (1964) work, Sisodia and Dwivedi (1981) suggested a modified ratio estimator for population mean of study variable by using the known CV of the auxiliary variable.

Later on Singh and Karan (1993), Upadhyaya and Singh (1999) proposed another ratio-type estimators with utilizing the known CV and coefficient of kurtosis (CK) of the auxiliary variable. All these authors have used the CV and CK of auxiliary variable in additive form to the sample and population means of the same variable.

Motivated by above points, in this work, an attempt has been made to estimate the population mean by utilizing the information on known coefficient of skewness (CS) and coefficient of kurtosis (CK) of the first auxiliary variable. We have also made use of our suggested estimators to construct some new chain ratio and chain regression type estimators and discussed its properties. The performances of these proposed estimators have been supported with a numerical illustration

1. Proposed Estimators

Let $\mathrm{U} = (\mathrm{u_1}, \mathrm{u_2}, \ldots, \mathrm{u_N})$ be the finite population of N units, y and x be the variables under study and first auxiliary variable respectively. It is assumed that y and x are highly positively correlated. Let $\mathrm{y_k} > 0$ and $\mathrm{x_k} > 0$ be the values of y and x for the k th $(\mathrm{k} = 1, 2, \ldots, N)$ unit in the population. From the population U, a simple random sample of size n is drawn without replacement. Let $(\overline{\mathrm{Y}}, \overline{\mathrm{X}})$ and $(\overline{\mathrm{y}}, \overline{\mathrm{x}})$ be the population means and sample means of the variable y and x respectively.

The classical ratio estimator for $\overline{\mathrm{Y}}$ is defined as

$$\overline{\mathrm{y}}_{\mathrm{r}} = \frac{\overline{\mathrm{y}}}{\overline{\mathrm{x}}}\overline{\mathrm{X}} \tag{1}$$

When x is negatively correlated with y , Robson (1964) proposed the product estimator of the population mean or total which is given below

$$\overline{\mathrm{y}}_{\mathrm{p}} = \frac{\overline{\mathrm{y}}\,\overline{\mathrm{x}}}{\overline{\mathrm{X}}} \tag{2}$$

This work was further developed by Singh, and Espejo (2003) as

$$\mathrm{t_{rp}} = \overline{\mathrm{y}} \left\{ \mathrm{k}\frac{\mathrm{X}}{\mathrm{x}} + (1-\mathrm{k})\frac{\overline{\mathrm{x}}}{\overline{\mathrm{X}}} \right\} \tag{3}$$

where k is a real constant to be determined such that, the mean squared error (M.S.E) of $\mathrm{t_{rp}}$ is minimum.

It may be noted that

for $\mathrm{k} = 1$, the estimator $\mathrm{t_{rp}}$ reduces to usual ratio estimator $\overline{\mathrm{y}}_\mathrm{r}$
and when $\mathrm{k} = 0$, it reduces to the usual product estimator $\overline{\mathrm{y}}_\mathrm{p}$.

If population mean of the auxiliary variable x i.e., $\overline{\mathrm{X}}$ is not known, we estimate $\overline{\mathrm{Y}}$ by two-phase ratio estimator

$$\mathrm{t_0} = \frac{\overline{\mathrm{y}}}{\overline{\mathrm{x}}}\overline{\mathrm{x}}' \tag{4}$$

where $\overline{\mathrm{x}}'$ is the sample mean of x based on the first-phase (preliminary) sample of size $\mathrm{n}'\,(\mathrm{n}' > \mathrm{n})$

which is further generalized by Srivastava (1970) as

$$\mathrm{t_1} = \overline{\mathrm{y}} \left(\frac{\overline{\mathrm{x}}'}{\overline{\mathrm{x}}} \right)^{a} \tag{5}$$

where α is determined so as to minimized the M.S.E of $\mathrm{t_1}$.

The way in which the estimate of $\overline{\mathrm{Y}}$ is improved using the auxiliary information on x can also be extended to improve the estimator of $\overline{\mathrm{X}}$ in the first-phase sample, if another auxiliary variable z closely related to x but remotely related to y is used. Thus, assuming that the population mean $\overline{\mathrm{Z}}$ of the variable z is known, Chand (1975) proposed a chain-type ratio estimator as

$$\mathrm{t_2} = \frac{\overline{\mathrm{y}}}{\overline{\mathrm{x}}}\overline{\mathrm{x}}'_{\mathrm{rd}}, \text{ where } \overline{\mathrm{x}}'_{\mathrm{rd}} = \frac{\overline{\mathrm{x}}'}{\overline{\mathrm{z}}'}\overline{\mathrm{Z}} \tag{6}$$

this technique was further modified by Srivastava (1990) as

$$\mathrm{t_3} = \overline{\mathrm{y}} \left(\frac{\overline{\mathrm{x}}'}{\overline{\mathrm{x}}} \right)^{\alpha_1} \left(\frac{\overline{\mathrm{Z}}}{\overline{\mathrm{z}}'} \right)^{\alpha_2} \tag{7}$$

where α_1, α_2 are suitably chosen constants to minimized the M.S.E of $\mathrm{t_3}$.

Kiregyera (1980) proposed a chain-type estimator

$$\mathrm{t_4} = \frac{\overline{\mathrm{y}}}{\overline{\mathrm{x}}}\overline{\mathrm{x}}'_{\mathrm{ld}}, \text{ where } \overline{\mathrm{x}}'_{\mathrm{ld}} = \overline{\mathrm{x}}' + \mathrm{b}'_{\mathrm{xz}} \left(\overline{\mathrm{Z}} - \overline{\mathrm{z}}' \right) \tag{8}$$

where b'_{xz} is the sample regression coefficient between x and z , based on sample of size n'.

Kiregyera (1984) proposed two chain-type estimators as

$$t_5 = \bar{y} + b_{yx}\left(\bar{x}'_{ld} - \bar{x}\right), \text{ where } \bar{x}'_{ld} = \bar{x}' + b'_{xz}\left(\bar{Z} - \bar{z}'\right) \tag{9}$$

and

$$t_6 = \bar{y}' + b_{yx}\left(\bar{x}'_{rd} - \bar{x}\right), \text{ where } \bar{x}'_{rd} = \frac{\bar{x}'}{\bar{z}'}\bar{Z} \tag{10}$$

where b_{yx} is sample regression coefficient between the y and x, based on sample of size n.
Utilizing the known coefficient of variation of z, Singh and Upadhyaya (1995) considered a modified chain-type ratio estimator as

$$t_7 = \frac{\bar{y}}{\bar{x}}\bar{x}'\left[\frac{\bar{Z} + C_z}{\bar{z}' + C_z}\right]^{\alpha} \tag{11}$$

where α is determined so as to minimized the M.S.E of t_7, and C_z is the known coefficient of variation of the variable z .

In many situations the values of the auxiliary variable may be available for each unit in the population, for instance, see Das and Tripathi (1981). In such situations the information on $\bar{Z}, C_z, \beta_1(z)$ (coefficient of skewness), $\beta_2(z)$ (coefficient of kurtosis) and possibly on some other parameters may be utilized. Regarding the availability of information on $\bar{Z}, C_z, \beta_1(z)$ and $\beta_2(z)$ researchers may be referred to Searls (1964), Sen (1978), Murthy (1967, pp. 96-99), Singh et al. (1973) and Searls and Intarpanich (1990). Using the known coefficient of variation C_z and known coefficient of kurtosis $\beta_2(z)$ of the second auxiliary variable z Upadhyaya and Singh (2001) considered the following estimators for $\bar{Y}$

$$t_8 = \frac{\bar{y}}{\bar{x}}\bar{x}'\left[\frac{\beta_2(z)\bar{Z} + C_z}{\beta_2(z)\bar{z}' + C_z}\right] \tag{12}$$

$$t_9 = \frac{\bar{y}}{\bar{x}}\bar{x}'\left[\frac{C_z\bar{Z} + \beta_2(z)}{C_z\bar{z}' + \beta_2(z)}\right] \tag{13}$$

Motivated by the above discussion, we have suggested a class of some modified ratio and product estimators of $\bar{Y}$, utilizing coefficient of skewness, coefficient of kurtosis, coefficient of variation of x as:

$$T_{ai} = \bar{y}\left\{k\frac{\bar{X} + g_i}{\bar{x} + g_i} + (1-k)\frac{\bar{x} + g_i}{\bar{X} + g_i}\right\}, (i = 1, 2, \ldots, 4) \tag{14}$$

Where k is suitable chosen constant to minimized the M.S.E of T_{ai}.

$\overline{X}, \overline{Y}$: population mean of x, y respectively .
$\overline{x}, \overline{y}$: sample means of the respective variables based on the sample of size n.

$$\begin{aligned} g_i &= 0 \text{ for } i = 1 \\ &= C_z \text{ (coefficient of variation) for } i = 2 \\ &= \beta_1(z) \text{ (coefficient of skewness) for } i = 3 \\ &= \beta_2(z) \text{ (coefficient of kurtosis) for } i = 4 \end{aligned}$$

It may be seen that $k = 1$ produces several ratio estimators for different values of g_i and for $g_i = 0$, it reduces to the usual ratio estimator.
Similarly, when $k = 0$, it reduces to several product estimators.

2. Bias and Mean Squared Errors (M.S.E) of the Proposed Estimators T_{ai}, $(i = 1, 2, \ldots, 4)$

As the proposed sequence of estimators $T_{ai}, (i = 1, 2, \ldots, 4)$ in (14) are biased for $\overline{Y}$, their biases and mean squared errors (M.S.E.'s) have been obtained up to the first order of approximations
$\overline{y} = \overline{Y}(1 + e_1), \overline{x} = \overline{X}(1 + e_2)$
and $E(e_1) = E(e_2) = 0$
The proposed sequence of estimators $T_{ai}(i = 1, 2, \ldots, 4)$ then become

$$T_{ai} = \overline{Y}(1 + e_1)\left\{1 + \theta_i e_2 + k\left(\theta_i^2 e_2^2 - 2\theta_i e_2\right)\right\} \tag{15}$$

Where $\theta_i = \frac{\overline{Z}}{\overline{Z}+g_i}, (i = 1, 2, \ldots 4)$

We expand the terms of (15) and collecting the terms up to the first order of approximation, we have the following results:

Theorem 2.1. *The bias* $B(.)$ *and mean squared error (M.S.E.'s)* $M(.)$ *of the proposed sequence of estimators* $T_{ai}(i = 1, 2, \ldots, 4)$*, to the terms of order of sample size i.e.,* $o\left(n^{-1}\right)$ *are derived as*

$$B(T_{ai}) = E\left(T_{ai} - \overline{Y}\right) = \overline{Y} f_1\left[\theta_i^2 C_x^2 + (1 - 2k)\theta_i \rho_{yx} C_y C_x\right] \tag{16}$$

$$M(T_{ai}) = E\left(T_{ai} - \overline{Y}\right)^2 = \overline{Y}^2 f_1\left\{\theta_i^2 (1 - 2k)^2 C_x^2 + C_y^2 + 2(1 - 2k)\theta_i \rho_{yx} C_y C_x\right\} \tag{17}$$

C_t *is the coefficient of variation of the variable* $t(t = x, y)$ *and* $f_1 = \left(\frac{1}{n} - \frac{1}{N}\right)$.

Corollary 2.1.1. *It is obvious that biases and M.S.E.'s of* $T_{ai}(i = 1, 2, \ldots, 4)$ *can be obtained by substituting the values of* $\theta_i(i = 1, 2, .., 4)$ *in (16 and 17) respectively.*

3. Minimum M.S.E of $\mathbf{T_{ai}, (i = 1, 2, .., 4)}$

The optimality condition, that is, the condition under which $\mathrm{T_{ai}}, (\mathrm{i} = 1, 2, \ldots, 4)$ has minimum M.S.E, is obtained as

$$\mathrm{k}_{(\text{opt })\mathrm{i}} = \frac{1}{2}\left(1 + \frac{\rho_{\mathrm{yx}}\mathrm{C_y}}{\theta_{\mathrm{i}}\mathrm{C_x}}\right) \tag{18}$$

Thus substituting this optimum value of k in equation (17), the minimum M.S.E of the class of estimators $\mathrm{T_{ai}}$, $(\mathrm{i} = 1, 2, \ldots, 4)$ is found as

$$\mathrm{M}\left(\ \mathrm{T_{ai}^0}\right) = \overline{\mathrm{Y}}^2 \mathrm{f_1}\left(1 - \rho_{\mathrm{yx}}^2\right)\mathrm{C_y^2}; (\mathrm{i} = 1, 2, \ldots, 4) \tag{19}$$

Remark 3.1. *The values of* $\mathrm{k}_{(opt)i}$, $(\mathrm{i} = 1, 2, \ldots, 4)$ *can be obtained by putting the values of is given by* $\theta_{\mathrm{i}}(\mathrm{i} = 1, 2, \ldots, 4$ *) respectively.*

4. Efficiency Comparisons of $\mathbf{T_{ai}(i = 1, 2, .., 4)}$

In this section, the conditions for which the proposed estimators are better than the other conventional estimators $\overline{\mathrm{y}}_{\mathrm{r}}$, $\overline{\mathrm{y}}_{\mathrm{p}}$ and $\mathrm{t_{rp}}$ have been demonstrated.
The M.S.E's of these estimators up to the first order of sample size are derived as

$$\mathrm{M}\left(\overline{\mathrm{y}}_{\mathrm{r}}\right) = \overline{\mathrm{Y}}^2 \mathrm{f_1}\left[\mathrm{C_y^2} + \mathrm{C_x^2} - 2\rho_{\mathrm{xy}}\mathrm{C_y}\mathrm{C_x}\right] \tag{20}$$

$$\mathrm{M}\left(\overline{\mathrm{y}}_{\mathrm{p}}\right) = \overline{\mathrm{Y}}^2 \mathrm{f_1}\left[\mathrm{C_y^2} + \mathrm{C_x^2} + 2\rho_{\mathrm{xy}}\mathrm{C_y}\mathrm{C_x}\right] \tag{21}$$

$$\min .\mathrm{M}\left(\mathrm{t_{rp}}\right) = \mathrm{f_1}\mathrm{S_y^2}\left(1 - \rho_{\mathrm{xy}}^2\right) \tag{22}$$

4.1. *From (17) and (20) we observe that*

$$\mathrm{M}\left(\mathrm{T_{ai}}\right) < \mathrm{M}\left(\overline{\mathrm{y}}_{\mathrm{r}}\right), \text{ if } \frac{-\mathrm{B} + \sqrt{\mathrm{B}^2 - 4\mathrm{AC}}}{2\mathrm{A}} < \theta_{\mathrm{i}} < \frac{-\mathrm{B} - \sqrt{\mathrm{B}^2 - 4\mathrm{AC}}}{2\mathrm{A}} \tag{23}$$

Where $\mathrm{A} = (1 - 2\mathrm{k})^2\mathrm{C_x^2},\ \mathrm{B} = 2(1 - 2\mathrm{k})\rho_{\mathrm{yx}}\mathrm{C_y}\mathrm{C_x},\ \mathrm{C} = 2\rho_{\mathrm{yx}}\mathrm{C_y}\mathrm{C_x} - \mathrm{C_x^2}$

4.2. *From (17) and (21) we can conclude that*

$$\mathrm{M}\left(\mathrm{T_{ai}}\right) < \mathrm{M}\left(\overline{\mathrm{y}}_{\mathrm{p}}\right), \text{ if } \frac{-\mathrm{B} + \sqrt{\mathrm{B}^2 - 4\mathrm{AC}}}{2\mathrm{A}} < \theta_{\mathrm{i}} < \frac{-\mathrm{B} - \sqrt{\mathrm{B}^2 - 4\mathrm{AC}}}{2\mathrm{A}} \tag{24}$$

Where $\mathrm{A} = (1 - 2\mathrm{k})^2\mathrm{C_x^2},\ \mathrm{B} = 2(1 - 2\mathrm{k})\rho_{\mathrm{yx}}\mathrm{C_y}\mathrm{C_x},\ \mathrm{C} = -\left(2\rho_{\mathrm{yx}}\mathrm{C_y}\mathrm{C_x} + \mathrm{C_x^2}\right)$

4.3. *From (17) and (22) we find that*

$$\mathrm{M}\left(\mathrm{T_{ai}}\right) < \mathrm{M}\left(\overline{\mathrm{y}}_{\mathrm{p}}\right), \text{ if } \frac{-\mathrm{B} + \sqrt{\mathrm{B}^2 - 4\mathrm{AC}}}{2\mathrm{A}} < \theta_{\mathrm{i}} < \frac{-\mathrm{B} - \sqrt{\mathrm{B}^2 - 4\mathrm{AC}}}{2\mathrm{A}} \tag{25}$$

Where $\mathrm{A} = (1-2\mathrm{k})^2\mathrm{C}_\mathrm{x}^2,\ \mathrm{B} = 2(1-2\mathrm{k})\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{x},\ \mathrm{C} = \overline{\mathrm{Y}}^2\rho_{\mathrm{yx}}^2\mathrm{C}_\mathrm{y}^2$

Remark 4.1. *Obviously* $\mathrm{B}^2 - 4\mathrm{AC} > 0$ *as* θ_i *is positive everywhere.*

5. Generalised class of estimators for $\overline{\mathbf{Y}}$ in single phase sampling

Motivated by Srivastava (1970), we have also formulated generalized version of the proposed class of estimators of population mean $\overline{\mathrm{Y}}$ as

$$\mathrm{T_{bi}} = \overline{\mathrm{y}}\left\{\mathrm{k}\frac{\overline{\mathrm{X}}+\mathrm{g_i}}{\overline{\mathrm{x}}+\mathrm{g_i}} + (1-\mathrm{k})\frac{\overline{\mathrm{x}}+\mathrm{g_i}}{\overline{\mathrm{X}}+\mathrm{g_i}}\right\}^{\alpha_1} \quad \text{for } \mathrm{i} = 1,2,\ldots,4 \tag{26}$$

Where α_1 is suitably chosen constant to minimized the M.S.Es of $\mathrm{T_{bi}}$.

6. Bias and Mean Squared Errors (M.S.E.'s) of the Proposed Estimators $\mathbf{T_{bi}}$ $(\mathbf{i = 1, 2, \ldots, 4}\ )$

As the proposed sequence of estimators $\mathrm{T_{bi}}(\mathrm{i} = 1,2,\ldots,4)$ in (26) are biased for $\overline{\mathrm{Y}}$, their biases and mean squared errors (M.S.E.'s) have been obtained up to the first order of approximations as:
Under the transformation indicated in section 2, the proposed sequence of estimators $\mathrm{T_{bi}}(\mathrm{i} = 1,2,\ldots,4)$ take the form

$$\mathrm{T_{bi}} = \overline{\mathrm{Y}}\left[\begin{array}{l} 1+\alpha_1\left\{\mathrm{k}\theta_\mathrm{i}^2\mathrm{e}_2^2 + (1-2\mathrm{k})\theta_\mathrm{i}\mathrm{e}_2\right\} + \frac{\alpha_1(\alpha_1-1)}{2}\left\{\theta_\mathrm{i}^2\mathrm{e}_2^2(1-2\mathrm{k})^2\right\} \\ +\mathrm{e}_1 + \alpha_1(1-2\mathrm{k})\theta_\mathrm{i}\mathrm{e}_1\mathrm{e}_2 \end{array}\right] \tag{27}$$

we expand the terms of (27) and collect the terms up to the first order of approximation, we have the following results.

Theorem 6.1. *The bias B (.) and mean squared error (M.S.E)* M(.) *of the proposed sequence of estimators* $\mathrm{T_{bi}}(\mathrm{i} = 1,2,\ldots,4)$, *to the terms of order* $\mathrm{o}\left(\mathrm{n}^{-1}\right)$ *are given by*

$$\mathrm{B}\left(\mathrm{T_{bi}}\right) = \mathrm{E}\left(\mathrm{T_{bi}} - \overline{\mathrm{Y}}\right) = \overline{\mathrm{Y}}\alpha_1\mathrm{f}_1\left[\begin{array}{l} \theta_\mathrm{i}^2\mathrm{C}_\mathrm{x}^2\left\{1+\left(\frac{\alpha_1-1}{2}\right)(1-2\mathrm{k})^2\right\} \\ +(1-2\mathrm{k})\theta_\mathrm{i}\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{x} \end{array}\right] \tag{28}$$

$$\mathrm{M}\left(\mathrm{T_{bi}}\right) = \mathrm{E}\left(\mathrm{T_{bi}} - \overline{\mathrm{Y}}\right)^2 = \overline{\mathrm{Y}}^2\mathrm{f}_1\left[\begin{array}{l} \alpha_1^2\theta_\mathrm{i}^2(1-2\mathrm{k})^2\mathrm{C}_\mathrm{x}^2 + \mathrm{C}_\mathrm{y}^2 \\ +2\alpha(1-2\mathrm{k})\theta_\mathrm{i}\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{x} \end{array}\right] \tag{29}$$

Corollary 6.1.1. *It is obvious that biases and M.S.E's of* $\mathrm{T_{bi}}\,(\mathrm{i} = 1,2,\ldots,4)$ *can be obtained by substituting the values of* $\theta_\mathrm{i}(\mathrm{i} = 1,2,\ldots,4\ )$ *in (28 and 29) respectively.*

7. Minimum M.S.E of $\mathbf{T_{bi}}$ $(\mathbf{i = 1, 2, \ldots, 4})$

The optimality condition, that is, the condition under which $T_{bi}, (i = 1, 2, \ldots, 4)$ has minimum M.S.E, is obtained as

$$\alpha_{1(opt)_i} = \frac{\rho_{yx}C_y}{(1-2k)\theta_i C_x} \tag{30}$$

Hence the minimum M.S.E of the class of estimators T_{bi}, $(i = 1, 2, \ldots, 4)$ is denoted by $M\left(T^0_{bi}\right)$ $(i = 1, 2, \ldots, 4)$ is given by

$$M\left(T^0_{bi}\right) = \overline{Y}^2 f_1\left(1-\rho^2_{yx}\right)C^2_y \tag{31}$$

Remark 7.1. *The values of* $\alpha_{1(opt)_i}(i = 1, 2, \ldots, 4)$ *can be obtained by putting the values of is given by* $\theta_i(i = 1, 2, .., 4)$ *respectively.*

8. Efficiency Comparisons of $\mathbf{T_{bi}}$ $(\mathbf{i = 1, 2, \ldots, 4})$

In this section, the conditions for which the proposed estimators are better than $\overline{y}_r$, $\overline{y}_p, t_{rp}$ and have been obtained. The M.S.E's of these estimators up to the order $o\left(n^{-1}\right)$ are derived as

8.1. *Comparison of* $\mathbf{T}_{bi}(\mathbf{i = 1, 2, \ldots, 4})$ *with* $\overline{\mathbf{y}}_{\mathbf{r}}$

It is obvious from (29) and (20) that $M(T_{bi}) < M(\overline{y}_r)$ $(i = 1, 2, \ldots, 4)$ if

$$\frac{-B+\sqrt{B^2-4AC}}{2A} < \alpha_1 < \frac{-B-\sqrt{B^2-4AC}}{2A} \tag{32}$$

Where

$$A = \overline{Y}^2 f_1\theta^2_i(1-2k)^2C^2_x, B = 2\overline{Y}^2 f_1\theta_i(1-2k)\rho_{yx}C_yC_x,$$
$$C = \overline{Y}^2 f_1\left(2\rho_{yx}C_yC_x - C^2_x\right)$$

8.2. *Comparison of* $\mathbf{T_{bi}}(\mathbf{i = 1, 2, \ldots, 4})$ *with* $\overline{\mathbf{y}}_{\mathbf{p}}$

It is obvious from (29) and (21) that $M(T_{bi}) < M\left(\overline{y}_p\right)$ if

$$\frac{-B+\sqrt{B^2-4AC}}{2A} < \alpha_1 < \frac{-B-\sqrt{B^2-4AC}}{2A} \tag{33}$$

Where

$$A = \overline{Y}^2 f_1 \theta_i^2 (1-2k)^2 C_x^2,\ B = 2\overline{Y}^2 f_1 \theta_i (1-2k) \rho_{yx} C_y C_x,$$
$$C = -\overline{Y}^2 f_1 \left(C_x^2 + 2\rho_{yx} C_y C_x \right)$$

8.3. *Comparison of* $\mathbf{T_{bi} (i = 1, 2, \dots, 4)}$ *with* $\mathbf{t_{rp}}$

It is obvious from (29) and (22) that $M(T_{bi}) < M(t_{rp})$ if

$$\frac{-B + \sqrt{B^2 - 4AC}}{2A} < \alpha_1 < \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{34}$$

Where

$$A = \overline{Y}^2 f_1 \theta_i^2 (1-2k)^2 C_x^2,\ B = 2\overline{Y}^2 f_1 \theta_i (1-2k) \rho_{yx} C_y C_x,$$
$$C = \overline{Y}^2 f_1 C_y^2 \rho_{yx}^2$$

Remark 8.1. *We have seen that* min. *M.S.E of* T_{bi} = min. *M.S.E of* T_{ai} = min. *M.S.E of* t_{rp}

9. SOME MODIFIED CHAIN-TYPE RATIO ESTIMATORS

Extending the above work in two phase sampling we define the following classes of modified chain-type ratio estimators for $\overline{Y}$

$$T_{ami} = \frac{\overline{y}}{\overline{x}} \overline{x}' \left\{ k \frac{\overline{Z} + g_i}{\overline{z}' + g_i} + (1-k) \frac{\overline{z}' + g_i}{\overline{Z} + g_i} \right\}^{\alpha_2} \quad \text{for } i = 1, 2, \dots, 4 \tag{35}$$

Where α_2 is determined so as to minimized the M.S.E of T_{ami} and
$\overline{Z}$: population mean of z.
$\overline{x}', \overline{z}'$: sample means of the respective variables of the sample of size n'.

9.1. *Bias and Mean Squared Errors (M.S.E.'s) of the Proposed Estimators* $\mathbf{T}_{ami}$, $\mathbf{(i = 1, 2, \dots, 4)}$

As the proposed sequence of estimators T_{ami} $(i = 1, 2, \dots, 4)$ in (35) are biased for $\overline{Y}$, their biases and mean squared errors (M.S.E.'s) have been obtained up to the first order of approximations.

$\overline{x}' = \overline{X}(1 + e_3)$, $\overline{z}' = \overline{Z}(1 + e_4)$
and $E(e_3) = E(e_4) = 0$.

The proposed sequence of estimators $T_{ami}(i = 1, 2, \dots, 4)$ takes the following forms

$$T_{ami} = \overline{Y}(1 + e_1)(1 + e_3)(1 + e_2)^{-1} \left\{ k(1 + \theta_i e_4)^{-1} + (1-k)(1 + \theta_i e_4) \right\}^{\alpha_2} \tag{36}$$

we expand the terms of (36) and collecting the terms up to the first order of approximation, we have the following results.

Theorem 9.1.1. *The bias* $\mathrm{B}(.)$ *and mean squared error (M.S.E.)* $\mathrm{M}(.)$ *of the proposed sequence of estimators* T_{ami} $(\mathrm{i}=1,2,\ldots,4)$, *to the terms of order* $\mathrm{o}\left(\mathrm{n}^{-1}\right)$ *are given by*

$\mathrm{B}(\mathrm{T}_{\mathrm{ami}})=\mathrm{E}\left(\mathrm{T}_{\mathrm{ami}}-\overline{\mathrm{Y}}\right)$

$$=\overline{\mathrm{Y}}\left[\begin{array}{l}\alpha_2\theta_{\mathrm{i}}^2\mathrm{f}_2\mathrm{C}_{\mathrm{z}}^2\left\{\mathrm{k}+\frac{(\alpha-1)}{2}(1-2\mathrm{k})^2\right\}+\alpha_2(1-2\mathrm{k})\theta_{\mathrm{i}}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{z}}\\-\left(\mathrm{f}_3\rho_{\mathrm{yx}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{x}}+\mathrm{f}_2\mathrm{C}_{\mathrm{x}}^2\right)\end{array}\right] \tag{37}$$

$$\mathrm{M}(\mathrm{T}_{\mathrm{ami}})=\mathrm{E}\left(\mathrm{T}_{\mathrm{ami}}-\overline{\mathrm{Y}}\right)^2=\overline{\mathrm{Y}}^2\left[\begin{array}{l}\alpha_2^2\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_{\mathrm{z}}^2+2\alpha_2(1-2\mathrm{k})\theta_{\mathrm{i}}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{z}}\\+\mathrm{f}_1\mathrm{C}_{\mathrm{y}}^2+\mathrm{f}_3\mathrm{C}_{\mathrm{x}}^2-2\mathrm{f}_3\rho_{\mathrm{yx}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{x}}\end{array}\right] \tag{38}$$

Where ρ_{yz} *the population coefficients of correlation between* y *and* z *used in suffixes,* C_{z} *is the coefficient of variation of* z, $\mathrm{f}_2=\left(\frac{1}{\mathrm{n}'}-\frac{1}{\mathrm{N}}\right)$, $\mathrm{f}_3=\left(\frac{1}{\mathrm{n}}-\frac{1}{\mathrm{n}'}\right)$

Corollary 9.1.1.1. *It is obvious that biases and M.S.E's of* T_{ami} $(\mathrm{i}=1,2,\ldots,4)$ *can be obtained by substituting the values of* $\theta_{\mathrm{i}}(\mathrm{i}=1,2,..,4.)$ *in (37 and 38) respectively.*

9.2. *Minimum M.S.E of* $\mathbf{T}_{ami}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$

The $\mathrm{M}(\mathrm{T}_{\mathrm{ami}})$ in (38) is minimized for

$$\alpha_{2(\mathrm{opt})\mathrm{i}}=\frac{\rho_{\mathrm{yz}}\mathrm{C}_{\mathrm{y}}}{(2\mathrm{k}-1)\theta_{\mathrm{i}}\mathrm{C}_{\mathrm{z}}} \tag{39}$$

Thus, the minimum M.S.E of the class of estimators $\mathrm{T}_{\mathrm{ami}}(\mathrm{i}=1,2,\ldots,4)$ is denoted by $\mathrm{M}\left(\mathrm{T}_{\mathrm{ami}}^{\mathrm{o}}\right),(\mathrm{i}=1,2,\ldots,4)$ is given by

$$\mathrm{M}\left(\mathrm{T}_{\mathrm{ami}}^0\right)=\overline{\mathrm{Y}}^2\left[\mathrm{f}_1\mathrm{C}_{\mathrm{y}}^2+\mathrm{f}_3\mathrm{C}_{\mathrm{x}}^2-2\mathrm{f}_3\rho_{\mathrm{yx}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{x}}-\mathrm{f}_2\rho_{\mathrm{yz}}^2\mathrm{C}_{\mathrm{y}}^2\right] \tag{40}$$

Remark 9.2.1. *The values of* $\alpha_{2(opt)\mathrm{i}}(\mathrm{i}=1,2,\ldots,4)$ *can be obtained by putting the values of is given by* $\theta_{\mathrm{i}}(\mathrm{i}=1,2,\ldots,4)$ *respectively.*

10. Efficiency Comparisons of $\mathbf{T}_{\mathrm{ami}}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$

In this section, the conditions for which the proposed sequence of estimators are better than the other estimators such as t_{i}, $(\mathrm{i}=1,2,\ldots,9)$ are derived.

The M.S.E.'s of these estimators up to the order $\mathrm{o}\left(\mathrm{n}^{-1}\right)$ are derived as

$$\mathrm{M}\left(\mathrm{t}_0\right)=\overline{\mathrm{Y}}^2\left[\mathrm{f}_1\mathrm{C}_\mathrm{y}^2+\mathrm{f}_3\left(\mathrm{C}_\mathrm{x}^2-2\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{x}\right)\right] \tag{41}$$

$$\min\mathrm{M}\left(\mathrm{t}_1\right)=\mathrm{S}_\mathrm{y}^2\left[\mathrm{f}_1-\mathrm{f}_3\rho_{\mathrm{xy}}^2\right] \tag{42}$$

$$\mathrm{M}\left(\mathrm{t}_2\right)=\overline{\mathrm{Y}}^2\left[\mathrm{f}_3\mathrm{C}_\mathrm{x}^2+\mathrm{f}_2\mathrm{C}_\mathrm{z}^2+\mathrm{f}_1\mathrm{C}_\mathrm{y}^2-2\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z}-2\mathrm{f}_3\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{x}\right] \tag{43}$$

$$\mathrm{M}\left(\mathrm{t}_3\right)=\mathrm{S}_\mathrm{y}^2\left[\mathrm{f}_1-\mathrm{f}_2\rho_{\mathrm{yz}}^2-\mathrm{f}_3\rho_{\mathrm{yx}}^2\right] \tag{44}$$

$$\mathrm{M}\left(\mathrm{t}_4\right)=\overline{\mathrm{Y}}^2\left[\mathrm{f}_1\mathrm{C}_\mathrm{y}^2+\mathrm{f}_3\mathrm{C}_\mathrm{x}^2-2\mathrm{f}_3\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{x}-2\mathrm{f}_2\rho_{\mathrm{yz}}\rho_{\mathrm{xz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{x}+\mathrm{f}_2\mathrm{C}_\mathrm{x}^2\rho_{\mathrm{xz}}^2\right] \tag{45}$$

$$\mathrm{M}\left(\mathrm{t}_5\right)=\overline{\mathrm{Y}}^2\left[\mathrm{f}_1\mathrm{C}_\mathrm{y}^2+\mathrm{f}_2\rho_{\mathrm{yx}}^2\mathrm{C}_\mathrm{y}^2\rho_{\mathrm{xz}}^2-\mathrm{f}_3\rho_{\mathrm{yx}}^2\mathrm{C}_\mathrm{y}^2-2\mathrm{f}_2\rho_{\mathrm{yx}}\rho_{\mathrm{yz}}\rho_{\mathrm{xz}}\mathrm{C}_\mathrm{y}^2\right] \tag{46}$$

$$\mathrm{M}\left(\mathrm{t}_6\right)=\overline{\mathrm{Y}}^2\left[\mathrm{f}_1\mathrm{C}_\mathrm{y}^2+\mathrm{f}_2\rho_{\mathrm{yx}}^2\frac{\mathrm{C}_\mathrm{y}^2\mathrm{C}_\mathrm{z}^2}{\mathrm{C}_\mathrm{x}^2}-\mathrm{f}_3\rho_{\mathrm{yx}}^2\mathrm{C}_\mathrm{y}^2-2\mathrm{f}_2\rho_{\mathrm{yx}}\rho_{\mathrm{yz}}\frac{\mathrm{C}_\mathrm{y}^2\mathrm{C}_\mathrm{z}}{\mathrm{C}_\mathrm{x}}\right] \tag{47}$$

$$\mathrm{M}\left(\mathrm{t}_7\right)=\overline{\mathrm{Y}}^2\left[\mathrm{f}_1\mathrm{C}_\mathrm{y}^2+\mathrm{f}_2\left(\psi^2\mathrm{C}_\mathrm{z}^2-2\psi\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z}\right)+\mathrm{f}_3\left(\mathrm{C}_\mathrm{x}^2-2\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{x}\right)\right] \tag{48}$$

$$\mathrm{M}\left(\mathrm{t}_8\right)=\overline{\mathrm{Y}}^2\left[\mathrm{f}_1\mathrm{C}_\mathrm{y}^2+\mathrm{f}_2\left(\mathrm{P}^2\mathrm{C}_\mathrm{z}^2-2\mathrm{P}\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z}\right)+\mathrm{f}_3\left(\mathrm{C}_\mathrm{x}^2-2\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{x}\right)\right] \tag{49}$$

$$\mathrm{M}\left(\mathrm{t}_9\right)=\overline{\mathrm{Y}}^2\left[\mathrm{f}_1\mathrm{C}_\mathrm{y}^2+\mathrm{f}_2\left(\mathrm{Q}^2\mathrm{C}_\mathrm{z}^2-2\mathrm{Q}\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z}\right)+\mathrm{f}_3\left(\mathrm{C}_\mathrm{x}^2-2\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{x}\right)\right] \tag{50}$$

10.1. *Comparison of* $\mathbf{T}_{ami}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_0}$

It is obvious from (38) and (41) that $\mathrm{M}\left(\mathrm{T}_{\mathrm{ami}}\right)<\mathrm{M}\left(\mathrm{t}_0\right)$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}}<\alpha_2<\frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{51}$$

Where

$$\mathrm{A}=\overline{\mathrm{Y}}^2\theta_\mathrm{i}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_\mathrm{z}^2,\ \mathrm{B}=2\overline{\mathrm{Y}}^2(1-2\mathrm{k})\theta_\mathrm{i}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z},$$
$$\mathrm{C}=0$$

10.2. *Comparison of* $\mathbf{T}_{ami}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{\min.M\left(t_1\right)}$

It is obvious from (38) and (42) that $\mathrm{M}\left(\mathrm{T}_{\mathrm{ami}}\right)<\min.\mathrm{M}\left(\mathrm{t}_1\right)$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}}<\alpha_2<\frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{52}$$

Where

$$\mathrm{A}=\overline{\mathrm{Y}}^2\theta_\mathrm{i}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_\mathrm{z}^2,\ \mathrm{B}=2\overline{\mathrm{Y}}^2(1-2\mathrm{k})\theta_\mathrm{i}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z},$$
$$\mathrm{C}=\overline{\mathrm{Y}}^2\mathrm{f}_3\left(\mathrm{C}_\mathrm{x}-\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\right)^2$$

10.3. *Comparison of* $\mathbf{T}_{ami}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_2}$

It is obvious from (38) and (43) that $\mathrm{M}\left(\mathrm{T_{ami}}\right) < \mathrm{M}\left(\mathrm{t_2}\right)$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_2 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{53}$$

Where

$$\begin{aligned}&\mathrm{A}=\overline{\mathrm{Y}}^2\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_{\mathrm{z}}^2,\ \mathrm{B}=2\overline{\mathrm{Y}}^2(1-2\mathrm{k})\theta_{\mathrm{i}}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{z}},\\&\mathrm{C}=\overline{\mathrm{Y}}^2\left(2\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{z}}-\mathrm{f}_2\mathrm{C}_{\mathrm{z}}^2\right)\end{aligned}$$

10.4. *Comparison of* $\mathbf{T}_{ami}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_3}$

It is obvious from (38) and (44) that $\mathrm{M}\left(\mathrm{T_{ami}}\right) < \mathrm{M}\left(\mathrm{t_3}\right)$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_2 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{54}$$

Where

$$\begin{aligned}&\mathrm{A}=\overline{\mathrm{Y}}^2\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_{\mathrm{z}}^2,\ \mathrm{B}=2\overline{\mathrm{Y}}^2(1-2\mathrm{k})\theta_{\mathrm{i}}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{z}},\\&\mathrm{C}=\overline{\mathrm{Y}}^2\left[\mathrm{f}_3\left(\mathrm{C}_{\mathrm{x}}-\rho_{\mathrm{yx}}\mathrm{C}_{\mathrm{y}}\right)^2+\mathrm{f}_2\rho_{\mathrm{yz}}^2\mathrm{C}_{\mathrm{y}}^2\right]\end{aligned}$$

10.5. *Comparison of* $\mathbf{T}_{ami}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_4}$

It is obvious from (38) and (45) that $\mathrm{M}\left(\mathrm{T_{ami}}\right) < \mathrm{M}\left(\mathrm{t_4}\right)$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_2 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{55}$$

Where

$$\begin{aligned}&\mathrm{A}=\overline{\mathrm{Y}}^2\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_{\mathrm{z}}^2,\ \mathrm{B}=2\overline{\mathrm{Y}}^2(1-2\mathrm{k})\theta_{\mathrm{i}}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{z}},\\&\mathrm{C}=\overline{\mathrm{Y}}^2\left[\mathrm{f}_2\left(\rho_{\mathrm{yz}}\rho_{\mathrm{xz}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{x}}-\mathrm{C}_{\mathrm{x}}^2\rho_{\mathrm{xz}}^2\right)\right]\end{aligned}$$

10.6. *Comparison of* $\mathbf{T}_{ami}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_5}$

It is obvious from (38) and (46) that $\mathrm{M}\left(\mathrm{T_{ami}}\right) < \mathrm{M}\left(\mathrm{t_5}\right)$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_2 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{56}$$

Where

$$\mathrm{A} = \overline{\mathrm{Y}}^2\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_\mathrm{z}^2,\ \mathrm{B} = 2\overline{\mathrm{Y}}^2(1-2\mathrm{k})\theta_\mathrm{i}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z},$$
$$\mathrm{C} = \overline{\mathrm{Y}}^2\left[\mathrm{f}_3\left(\mathrm{C}_\mathrm{x}-\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\right)^2 + \mathrm{f}_2\left(\rho_{\mathrm{yx}}^2\rho_{\mathrm{xz}}^2 - 2\rho_{\mathrm{yx}}\rho_{\mathrm{yz}}\rho_{\mathrm{xz}}\right)\mathrm{C}_\mathrm{y}^2\right]$$

10.7. *Comparison of* $\mathbf{T}_{ami}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_6}$

It is obvious from (38) and (47) that $\mathrm{M}(\mathrm{T_{ami}}) < \mathrm{M}(\mathrm{t_6})$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_2 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{57}$$

Where

$$\mathrm{A} = \overline{\mathrm{Y}}^2\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_\mathrm{z}^2,\ \mathrm{B} = 2\overline{\mathrm{Y}}^2(1-2\mathrm{k})\theta_\mathrm{i}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z},$$
$$\mathrm{C} = \overline{\mathrm{Y}}^2\left[\mathrm{f}_3\left(\mathrm{C}_\mathrm{x}^2-\rho_{\mathrm{yx}}\mathrm{C}_\mathrm{y}\right)^2 + \mathrm{f}_2\left(2\rho_{\mathrm{yx}}\rho_{\mathrm{yz}}\frac{\mathrm{C}_\mathrm{y}^2\mathrm{C}_\mathrm{z}}{\mathrm{C}_\mathrm{x}} - \rho_{\mathrm{yx}}^2\frac{\mathrm{C}_\mathrm{y}^2\mathrm{C}_\mathrm{z}^2}{\mathrm{C}_\mathrm{x}^2}\right)\right]$$

10.8. *Comparison of* $\mathbf{T}_{ami}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_7}$

It is obvious from (38) and (48) that $\mathrm{M}(\mathrm{T_{ami}}) < \mathrm{M}(\mathrm{t_7})$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_2 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{58}$$

Where

$$\mathrm{A} = \overline{\mathrm{Y}}^2\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_\mathrm{z}^2,\ \mathrm{B} = 2\overline{\mathrm{Y}}^2(1-2\mathrm{k})\theta_\mathrm{i}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z},$$
$$\mathrm{C} = \overline{\mathrm{Y}}^2\left[\mathrm{f}_2\left(2\psi\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z} - \psi^2\mathrm{C}_\mathrm{z}^2\right)\right]$$

10.9. *Comparison of* $\mathbf{T}_{ami}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_8}$

It is obvious from (38) and (49) that $\mathrm{M}(\mathrm{T_{ami}}) < \mathrm{M}(\mathrm{t_8})$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_2 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{59}$$

Where

$$\mathrm{A} = \overline{\mathrm{Y}}^2\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_\mathrm{z}^2,\ \mathrm{B} = 2\overline{\mathrm{Y}}^2(1-2\mathrm{k})\theta_\mathrm{i}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z},$$
$$\mathrm{C} = \overline{\mathrm{Y}}^2\mathrm{f}_2\left(2\mathrm{P}\rho_{\mathrm{yz}}\mathrm{C}_\mathrm{y}\mathrm{C}_\mathrm{z} - \mathrm{P}^2\mathrm{C}_\mathrm{z}^2\right)$$

10.10. *Comparison of* $\mathbf{T}_{ami}$, $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{4})$ *with* $\mathbf{t}_9$

It is obvious from (38) and (50) that $\mathrm{M}(\mathrm{T}_{\mathrm{ami}}) < \mathrm{M}(\mathrm{t}_9)$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_2 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{60}$$

Where

$$\mathrm{A} = \overline{\mathrm{Y}}^2\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2\mathrm{f}_2\mathrm{C}_{\mathrm{z}}^2,\ \mathrm{B} = 2\overline{\mathrm{Y}}^2(1-2\mathrm{k})\theta_{\mathrm{i}}\mathrm{f}_2\rho_{\mathrm{yz}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{z}},$$
$$\mathrm{C} = \overline{\mathrm{Y}}^2\mathrm{f}_2\left(2\mathrm{Q}\rho_{\mathrm{yz}}\mathrm{C}_{\mathrm{y}}\mathrm{C}_{\mathrm{z}} - \mathrm{Q}^2\mathrm{C}_{\mathrm{z}}^2\right)$$

11. SOME MODIFIED CHAIN-TYPE REGRESSION ESTIMATORS

Similarly, we define the following classes of modified chain-type regression estimators for $\overline{\mathrm{Y}}$

$$\mathrm{T}_{\mathrm{armi}} = \overline{\mathrm{y}} + \mathrm{b}_{\mathrm{yx}}\left[\overline{\mathrm{x}}'\left\{\mathrm{k}\frac{\overline{\mathrm{Z}}+\mathrm{g}_{\mathrm{i}}}{\overline{\mathrm{z}}'+\mathrm{g}_{\mathrm{i}}} + (1-\mathrm{k})\frac{\overline{\mathrm{z}}'+\mathrm{g}_{\mathrm{i}}}{\overline{\mathrm{Z}}+\mathrm{g}_{\mathrm{i}}}\right\}^{\alpha_3} - \overline{\mathrm{x}}\right] \text{ for } \mathrm{i} = 1, 2, \ldots, 4 \tag{61}$$

Where α_3 is determined so as to minimized the M.S.E of $\mathrm{T}_{\mathrm{armi}}$.

12. Bias and Mean Squared Errors (M.S.E.'s) of the Proposed Estimators $\mathbf{T}_{\mathbf{armi}}$, $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{4})$

As the proposed sequence of estimators $\mathrm{T}_{\mathrm{armi}}(\mathrm{i} = 1, 2, \ldots, 4)$ in (61) are biased for $\overline{\mathrm{Y}}$, their biases and mean squared errors(M.S.E's) have been obtained up to the first order of approximations

$\mathrm{s}_{\mathrm{yx}} = \mathrm{S}_{\mathrm{yx}}(1+\mathrm{e}_5)$, $\mathrm{s}_{\mathrm{x}}^2 = \mathrm{S}_{\mathrm{x}}^2(1+\mathrm{e}_6)$
and $\mathrm{E}(\mathrm{e}_5) = \mathrm{E}(\mathrm{e}_6) = 0$.

The proposed sequence of estimators $\mathrm{T}_{\mathrm{armi}}(\mathrm{i} = 1, 2, \ldots, 4)$ then become

$$\mathrm{T}_{\mathrm{armi}} = \overline{\mathrm{Y}}(1+\mathrm{e}_1) + \beta_{yx}\left(1-\mathrm{e}_6+\mathrm{e}_6^2+\mathrm{e}_5-\mathrm{e}_5\mathrm{e}_6\right)\left[\overline{\mathrm{X}}\left\{\mathrm{e}_3-\mathrm{e}_2+\alpha_3\theta_{\mathrm{i}}\mathrm{e}_4(1-2\mathrm{k})\right\}\right] \tag{62}$$

Where α_3 is population regression between the variable y and x.

We expand the terms of (62) and collecting the terms up to the first order of approximation, we have the following results:

Theorem 12.1. *The bias* $\mathrm{B}(.)$ *and mean squared error* $\mathrm{M}(.)$ *of the proposed sequence of estimators* T_{armi} $(\mathrm{i} = 1, 2, \ldots, 4)$ *to the terms of order* $\mathrm{o}\left(\mathrm{n}^{-1}\right)$ *are given by*

$$B\left(\mathrm{T_{armi}}\right) = E\left(\mathrm{T_{armi}} - \overline{\mathrm{Y}}\right) =$$

$$\beta_{yx}\overline{X}\left[\begin{array}{l} \alpha_3\theta_i^2 f_2 C_z^2\left\{k + \frac{(\alpha-1)}{2}(1-2k)^2\right\} + \alpha(1-2k)\theta_i f_2 \rho_{xz} C_x C_z \\ +\alpha(1-2k)\theta_i f_2\left(\frac{\mu_{1,1,1}}{\overline{Z}\ S_{yx}} - \frac{\mu_{2,0,1}}{\overline{Z}\ S_x^2}\right) - f_3\left(\frac{\mu_{2,1,0}}{\overline{X}\ S_{yx}} - \frac{\mu_{3,0,0}}{\overline{X}\ S_x^2}\right) \end{array}\right] \tag{63}$$

Where $\mu_{rst} = E\left[\left(x_i - \overline{X}\right)^r \left(y_i - \overline{Y}\right)^s \left(z_i - \overline{Z}\right)^t\right];\ r \geq 0,\ s \geq 0,\ t \geq 0$

$$M\left(T_{armi}\right) = E\left(T_{armi} - \overline{Y}\right)^2 =$$

$$\overline{Y}^2\left[\begin{array}{l} f_1 C_y^2 + 2f_2\rho_{yx}\rho_{yz}\frac{C_y^2 C_z}{C_x}\alpha_3\theta_i(1-2k) \\ -f_3\rho_{yx}^2 C_y^2 + f_2\rho_{yx}^2\frac{C_y^2 C_z^2}{C_x^2}\alpha_3^2\theta_i^2(1-2k)^2 \end{array}\right] \tag{64}$$

Where ρ_{yx} *and* ρ_{xz} *the population coefficients of correlation between the variables shown in suffixes.*

Corollary 12.1.1. *It is obvious that biases and M.S.E.'s of* T_{armi} $(i = 1, 2, \ldots, 4)$ *can be obtained by substituting the values of* $\theta_i (i = 1, 2, .., 4\ )$ *in (63 and 64) respectively.*

13. Minimum M.S.E of $\mathbf{T_{armi}}, (\mathbf{i = 1, 2, \ldots, 4})$

The $M\left(T_{armi}\right)$ in () is minimized for

$$\alpha_{3(opt)i} = \frac{\rho_{yz}C_x}{(2k-1)\theta_i\rho_{yx}C_z} \tag{65}$$

Hence the minimum M.S.E of the class of estimators $T_{armi}(i = 1, 2, \ldots, 4)$ is denoted by $M\left(T^o_{armi}\ \right), (i = 1, 2, \ldots, 4)$ is given by

$$M\left(T^0_{armi}\ \right) = \overline{Y}^2\left[f_1 C_y^2 - f_3\rho_{yx}^2 C_y^2 - f_2\rho_{yz}^2 C_y^2\right] \tag{66}$$

Remark 13.1. *The values of* $\alpha_{3(opt)i}(i = 1, 2, \ldots, 4)$ *can be obtained by substituting the values of is given by* $\theta_i(i = 1, 2, \ldots, 4)$ *respectively.*

14. Efficiency Comparisons of $\mathbf{T_{armi}}, (\mathbf{i = 1, 2, \ldots, 4})$

In this section, the conditions for which the proposed class of estimators are better than the other estimators such as $t_i, (i = 1, 2, \ldots, 9)$ are derived. The M.S.E's of these estimators up to the order $o\left(n^{-1}\right)$ are derived as

14.1. *Comparison of* $\mathbf{T_{armi}}, (\mathbf{i = 1, 2, \ldots, 4})$ *with* $\mathbf{t_0}$

It is obvious from (64) and (41) that $M\left(T_{armi}\right) < M\left(t_0\right)$

$$\frac{-B + \sqrt{B^2 - 4AC}}{2A} < \alpha_3 < \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{67}$$

Where

$$\mathrm{A}=\overline{\mathrm{Y}}^{2}\mathrm{f}_2\rho_{\mathrm{yx}}^2\frac{\mathrm{C_y^2C_z^2}}{\mathrm{C_x^2}}\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2,\ \mathrm{B}=2\overline{\mathrm{Y}}^{2}\mathrm{f}_2\rho_{\mathrm{yx}}\rho_{\mathrm{yz}}\frac{\mathrm{C_y^2C_z}}{\mathrm{C_x}}\theta_{\mathrm{i}}(1-2\mathrm{k}),$$
$$\mathrm{C}=2\overline{\mathrm{Y}}^{2}\mathrm{f}_3\left(\rho_{\mathrm{yx}}\mathrm{C_y}\mathrm{C_x}-\mathrm{C_x^2}\right)$$

14.2. *Comparison of* $\mathbf{T}_{armi}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_1}$

It is obvious from (64) and (42) that $\mathrm{M}\left(\mathrm{T_{armi}}\right)<\min.\mathrm{M}\left(\mathrm{t_1}\right)$ if

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}}<\alpha_3<\frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{68}$$

Where

$$\mathrm{A}=\overline{\mathrm{Y}}^{2}\mathrm{f}_2\rho_{\mathrm{yx}}^2\frac{\mathrm{C_y^2C_z^2}}{\mathrm{C_x^2}}\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2,\ \mathrm{B}=2\overline{\mathrm{Y}}^{2}\mathrm{f}_2\rho_{\mathrm{yx}}\rho_{\mathrm{yz}}\frac{\mathrm{C_y^2C_z}}{\mathrm{C_x}}\theta_{\mathrm{i}}(1-2\mathrm{k}),$$
$$\mathrm{C}=0$$

14.3. *Comparison of* $\mathbf{T}_{armi}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_2}$

It is obvious from (64) and (43) that $\mathrm{M}\left(\mathrm{T_{armi}}\right)<\mathrm{M}\left(\mathrm{t_2}\right)$

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}}<\alpha_3<\frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{69}$$

Where

$$\mathrm{A}=\overline{\mathrm{Y}}^{2}\mathrm{f}_2\rho_{\mathrm{yx}}^2\frac{\mathrm{C_y^2C_z^2}}{\mathrm{C_x^2}}\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2,\ \mathrm{B}=2\overline{\mathrm{Y}}^{2}\mathrm{f}_2\rho_{\mathrm{yx}}\rho_{\mathrm{yz}}\frac{\mathrm{C_y^2C_z}}{\mathrm{C_x}}\theta_{\mathrm{i}}(1-2\mathrm{k}),$$
$$\mathrm{C}=\overline{\mathrm{Y}}^{2}\mathrm{f}_2\rho_{\mathrm{yz}}^2$$

14.4. *Comparison of* $\mathbf{T}_{armi}$, $(\mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{4})$ *with* $\mathbf{t_3}$

It is obvious from (64) and (44) that $\mathrm{M}\left(\mathrm{T_{armi}}\right)<\mathrm{M}\left(\mathrm{t_3}\right)$

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}}<\alpha_3<\frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{70}$$

Where

$$\mathrm{A}=\overline{\mathrm{Y}}^{2}\mathrm{f}_2\rho_{\mathrm{yx}}^2\frac{\mathrm{C_y^2C_z^2}}{\mathrm{C_x^2}}\theta_{\mathrm{i}}^2(1-2\mathrm{k})^2,\ \mathrm{B}=2\overline{\mathrm{Y}}^{2}\mathrm{f}_2\rho_{\mathrm{yx}}\rho_{\mathrm{yz}}\frac{\mathrm{C_y^2C_z}}{\mathrm{C_x}}\theta_{\mathrm{i}}(1-2\mathrm{k}),$$
$$\mathrm{C}=\overline{\mathrm{Y}}^{2}\mathrm{f}_2\rho_{\mathrm{yz}}^2$$

14.5. *Comparison of* $\mathbf{T}_{armi}$, $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{4})$ *with* $\mathbf{t_4}$

It is obvious from (64) and (45) that $\mathrm{M}\left(\mathrm{T_{armi}}\right) < \mathrm{M}\left(\mathrm{t_4}\right)$

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_3 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{71}$$

Where

$$\mathrm{A} = \overline{\mathrm{Y}}^2 \mathrm{f_2} \rho_{\mathrm{yx}}^2 \frac{\mathrm{C_y^2 C_z^2}}{\mathrm{C_x^2}} \theta_{\mathrm{i}}^2 (1-2\mathrm{k})^2,\ \mathrm{B} = 2\overline{\mathrm{Y}}^2 \mathrm{f_2} \rho_{\mathrm{yx}} \rho_{\mathrm{yz}} \frac{\mathrm{C_y^2 C_z}}{\mathrm{C_x}} \theta_{\mathrm{i}} (1-2\mathrm{k}),$$
$$\mathrm{C} = \overline{\mathrm{Y}}^2 \left[\mathrm{f_2}\left(2\rho_{\mathrm{yz}}\rho_{\mathrm{xz}}\mathrm{C_y C_x} - \mathrm{C_x^2}\rho_{\mathrm{xz}}^2\right) - \mathrm{f_3}\left(\mathrm{C_x} - \rho_{\mathrm{yx}}\mathrm{C_y}\right)^2\right]$$

14.6. *Comparison of* $\mathbf{T}_{armi}$, $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{4})$ *with* $\mathbf{t_5}$

It is obvious from (64) and (46) that $\mathrm{M}\left(\mathrm{T_{armi}}\right) < \mathrm{M}\left(\mathrm{t_5}\right)$

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_3 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{72}$$

Where

$$\mathrm{A} = \overline{\mathrm{Y}}^2 \mathrm{f_2} \rho_{\mathrm{yx}}^2 \frac{\mathrm{C_y^2 C_z^2}}{\mathrm{C_x^2}} \theta_{\mathrm{i}}^2 (1-2\mathrm{k})^2,\ \mathrm{B} = 2\overline{\mathrm{Y}}^2 \mathrm{f_2} \rho_{\mathrm{yx}} \rho_{\mathrm{yz}} \frac{\mathrm{C_y^2 C_z}}{\mathrm{C_x}} \theta_{\mathrm{i}} (1-2\mathrm{k}),$$
$$\mathrm{C} = \overline{\mathrm{Y}}^2 \left[\mathrm{f_2}\left(2\rho_{\mathrm{yz}} - \rho_{\mathrm{yx}}\rho_{\mathrm{xz}}\right)\rho_{\mathrm{yx}}\rho_{\mathrm{xz}}\mathrm{C_y^2}\right]$$

14.7. *Comparison of* $\mathbf{T}_{armi}$, $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{4})$ *with* $\mathbf{t_6}$

It is obvious from (64) and (47) that $\mathrm{M}\left(\mathrm{T_{armi}}\right) < \mathrm{M}\left(\mathrm{t_6}\right)$

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_3 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{73}$$

Where

$$\mathrm{A} = \overline{\mathrm{Y}}^2 \mathrm{f_2} \rho_{\mathrm{yx}}^2 \frac{\mathrm{C_y^2 C_z^2}}{\mathrm{C_x^2}} \theta_{\mathrm{i}}^2 (1-2\mathrm{k})^2,\ \mathrm{B} = 2\overline{\mathrm{Y}}^2 \mathrm{f_2} \rho_{\mathrm{yx}} \rho_{\mathrm{yz}} \frac{\mathrm{C_y^2 C_z}}{\mathrm{C_x}} \theta_{\mathrm{i}} (1-2\mathrm{k}),$$
$$\mathrm{C} = \overline{\mathrm{Y}}^2 \left[\mathrm{f_2}\left(2\rho_{\mathrm{yz}} - \frac{\rho_{\mathrm{yx}}}{\mathrm{C_x}}\right)\rho_{\mathrm{yx}}\frac{\mathrm{C_y^2 C_z}}{\mathrm{C_x}}\right]$$

14.8. *Comparison of* $\mathbf{T}_{armi}$, $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{4})$ *with* $\mathbf{t_7}$

It is obvious from (64) and (48) that $\mathrm{M}\left(\mathrm{T_{armi}}\right) < \mathrm{M}\left(\mathrm{t_7}\right)$

$$\frac{-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} < \alpha_3 < \frac{-\mathrm{B}-\sqrt{\mathrm{B}^2-4\mathrm{AC}}}{2\mathrm{A}} \tag{74}$$

Where

$$A = \overline{Y}^2 f_2 \rho_{yx}^2 \frac{C_y^2 C_z^2}{C_x^2} \theta_i^2 (1-2k)^2,\ B = 2\overline{Y}^2 f_2 \rho_{yx} \rho_{yz} \frac{C_y^2 C_z}{C_x} \theta_i (1-2k),$$
$$C = \overline{Y}^2 \left[f_2 \left(2\psi \rho_{yz} C_y C_z - \psi^2 C_z^2 \right) - f_3 \left(C_x - \rho_{yx} C_y \right)^2 \right]$$

14.9. *Comparison of* $\mathbf{T}_{armi}$, $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{4})$ *with* $\mathbf{t_8}$

It is obvious from (64) and (49) that $M(T_{armi}) < M(t_8)$

$$\frac{-B + \sqrt{B^2 - 4AC}}{2A} < \alpha_3 < \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{75}$$

Where

$$A = \overline{Y}^2 f_2 \rho_{yx}^2 \frac{C_y^2 C_z^2}{C_x^2} \theta_i^2 (1-2k)^2,\ B = 2\overline{Y}^2 f_2 \rho_{yx} \rho_{yz} \frac{C_y^2 C_z}{C_x} \theta_i (1-2k),$$
$$C = \overline{Y}^2 \left[f_2 \left(2P \rho_{yz} C_y C_z - P^2 C_z^2 \right) - f_3 \left(C_x - \rho_{yx} C_y \right)^2 \right]$$

14.10. *Comparison of* $\mathbf{T}_{armi}$, $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{4})$ *with* $\mathbf{t_9}$

It is obvious from (64) and (50) that $M(T_{armi}) < M(t_9)$

$$\frac{-B + \sqrt{B^2 - 4AC}}{2A} < \alpha_3 < \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{76}$$

Where

$$A = \overline{Y}^2 f_2 \rho_{yx}^2 \frac{C_y^2 C_z^2}{C_x^2} \theta_i^2 (1-2k)^2,\ B = 2\overline{Y}^2 f_2 \rho_{yx} \rho_{yz} \frac{C_y^2 C_z}{C_x} \theta_i (1-2k),$$
$$C = \overline{Y}^2 \left[f_2 \left(2Q \rho_{yz} C_y C_z - Q^2 C_z^2 \right) - f_3 \left(C_x - \rho_{yx} C_y \right)^2 \right]$$

15. Comparison of min. M.S.E of $\mathbf{T_{armi}}$ with min. M.S.E of $\mathbf{T_{ami}}$

We observe from (66) and (40) that $\min.M(T_{armi}) < \min.M(T_{ami})$ if

$$f_3 \left(C_x - \rho_{yx} C_y \right)^2 > 0 \tag{77}$$

which is always positive as f_3 is always positive.

16. Numerical Illustrations

We consider the data used by Anderson (1958) to demonstrate the merits of the proposed strategies. 25 families have been observed for the following three variables.

y : Head length of second son
x : Head length of first son
z : Head breadth of first son

$\overline{Y} = 183.84, \overline{X} = 185.72, \overline{Z} = 151.12, R(x) = 34, R(z) = 30, C_y = 0.0546, C_x = 0.0526, C_z = 0.0488, \rho_{yx} = 0.7108, \rho_{yz} = 0.6932, \rho_{xz} = 0.7346, \beta_1(z) = 0.0002, \beta_2(z) = 2.6519$, Consider $n = 7, n' = 10, \beta_{yx.z} = 0.4499, \beta_{yz.x} = 0.5057, \rho_{y.xz} = 0.7542$

We have computed percent relative efficiency (PREs) of the conventional estimators of population mean along with the proposed estimators and displayed in Table 1.

Table 1.

ESTIMATORS	$\bar{y}$	$\bar{y}_r$	$\bar{y}_p$	t_{rp}	$\bar{y}^o_{ai}$ $\forall i = 1, 2, .., 4$	T^0_{ai} $\forall i = 1, 2, \ldots, 4$
MSE/min MSE	10.3633	5.7885	34.1742	5.1274	5.1274	5.1274
PRE	100	179.03	30.3249	202.12	202.12	202.12

We have seen that our proposed estimators i.e $\bar{y}_{ai}$ and T_{ai} are preferable over $\bar{y}_r$ and $\bar{y}_p$ under optimum condition.
If we assume that mean of X is unknown then we have to go for two phase sampling. Here we have presented a comparable study of our proposed estimators with other considered estimator in two phase sampling. We have computed M.S.E.'s of the different proposed estimators and percent relative efficiency (PRE) of different estimators of $\overline{Y}$ in two phase sampling with respect to usual unbiased estimator of $\bar{y}$ and compiled in Table 2.

Table 2.

ESTIMATORS	$\mathbf{T^o_{ami}}$ $\forall i = 1, 2, \ldots, 4$	$\mathbf{T^o_{armi}}$ $\forall i = 1, 2, \ldots, 4$	$\mathbf{t_0}$	$\mathbf{t_1}$
MSE/min MSE/variance	5.5511	5.2757	8.4573	8.1818
PRE	186.69	196.43	122.54	126.66
ESTIMATORS	t_2	t_3	t_4	t_5
MSE/min MSE	5.7962	5.2769	5.5427	5.4751
PRE	178.80	196.39	186.97	189.28
ESTIMATORS	t_6	t_7	t_8	t_9
MSE/min MSE	5.2837	5.7955	5.7959	5.5617
PRE	196.14	178.82	178.80	186.33

From Table2. we observed that our suggested estimators $T_{ami}(i = 1, 2, \ldots, 4)$ and T_{armi} $(i = 1, 2, \ldots, 4)$ are better than other considered estimators under optimum condition. We have also verified that $T_{armi}, (i = 1, 2, \ldots, 4)$ is better than $T_{ami}, (i = 1, 2, \ldots, 4)$ under optimum conditions. Therefore, our suggested estimators are more justifiable in compare with the previous work of similar nature.

17. Conclusion

It is found that the estimators which are proposed in this manuscript are better than other conventional estimators of population mean. Therefore, the use of these estimators may be recommended to the survey statisticians.

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